Magnetic properties and variational calculus

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The finite element method is currently the mainstream method in the field of low frequency electromagnetic field analysis. In this method, Maxwell's equation, which is a fundamental equation, is formulated using a weighted residual method such as the Galerkin method.　This is because the magnetization characteristics of magnetic materials are generally not linear with constant magnetic permeability, but have a nonlinear dependence on magnetic flux density.

In the case of a linear magnetic material, the fundamental equation can be expressed as follows using the variational calculus.

$$\delta \int_V [\frac{1}{2\mu}(\mathrm{rot}\boldsymbol{A})^2 - \boldsymbol{J}\cdot\boldsymbol{A}]dV = 0 \tag{1}$$

Here μ is permeability and has a constant value.　***A*** is vector potential, and ***J*** is current density.　Taking variations, the left-hand side becomes as follows.

$$\begin{aligned}&\int_V [\frac{1}{\mu}\mathrm{rot}\delta\boldsymbol{A}\cdot\mathrm{rot}\boldsymbol{A} - \delta\boldsymbol{A}\cdot\boldsymbol{J}]dV \\ &= \int_V \delta\boldsymbol{A}\cdot[\mathrm{rot}\frac{1}{\mu}\mathrm{rot}\boldsymbol{A} - \boldsymbol{J}]dV + \text{(Surface integral)}\end{aligned} \tag{2}$$

However, since the transformation of the last formula was performed using Gauss' theorem, a surface integral has appeared. Since this surface integral normally disappears through boundary conditions, it is required that the integral of the first term be zero, and it is possible to solve the electromagnetic field equation by the variational calculus.

However, in general magnetic materials, the magnetic permeability is not constant, and so such variational calculus cannot be used.　Therefore, when dealing with these kinds of magnetic materials using the finite element method, we utilize the vector weighting function ***W*** to produce the following equation.

$$\int_V \boldsymbol{W}\cdot[\mathrm{rot}\frac{1}{\mu}\mathrm{rot}\boldsymbol{A} - \boldsymbol{J}]dV = 0 \tag{3}$$

If the left-hand side can be transformed using Gauss' theorem and the surface integral eliminated through boundary conditions, the formula becomes as follows.

$$\int_V [\frac{1}{\mu}\mathrm{rot}\boldsymbol{W}\cdot\mathrm{rot}\boldsymbol{A} - \boldsymbol{W}\cdot\boldsymbol{J}]dV = 0 \tag{4}$$

In this study, we show that the variational calculus can be used even for general magnetic materials by considering the thermodynamics of the magnetic material, and demonstrate that, in electromagnetic field analysis also, the finite element method can be formulated naturally.

Considering the free energy *F* of the magnetic material as a function of temperature *T* and magnetic flux density ***B***, this differential can be expressed as follows.

$$dF(T,\boldsymbol{B}) = -SdT + \boldsymbol{H}\cdot d\boldsymbol{B} \tag{5}$$

Here, *S* is the entropy of the magnetic material per unit volume, *T* is temperature, and ***H*** is magnetic field.　From this, the thermodynamic variables can be expressed as follows.

$$\begin{aligned}S &= -\frac{\partial}{\partial T}F(T,\boldsymbol{B}) \\ \boldsymbol{H} &= \frac{\partial}{\partial \boldsymbol{B}}F(T,\boldsymbol{B})\end{aligned} \tag{6}$$

Here we introduce the following thermodynamic potential by transforming variables.

$$G(T,\boldsymbol{H}) = F(T,\boldsymbol{B}) - \boldsymbol{H}\cdot\boldsymbol{B} \tag{7}$$

Calculating this derivative, the following is obtained from Eq. (5).

$$dG(T,\boldsymbol{H}) = -SdT - \boldsymbol{B}\cdot d\boldsymbol{H} \tag{8}$$

In electromagnetic field analysis, the magnetic field is often obtained by inputting a current, which corresponds to the problem of finding the magnetic flux density for a magnetic field $\boldsymbol{H}$ generated by an electric current. According to thermodynamics, for fixed temperature and magnetic field, this temperature thermodynamic potential is at its minimum at equilibrium. Therefore, the variation of the following integral must be zero if temperature is constant.

$$\delta\int_V G(T,\boldsymbol{H})dV = 0 \tag{9}$$

Since temperature and magnetic field are here assumed to be fixed, this variation is taken on magnetic flux density, which is the other state quantity. The variation on the left-hand side of this equation is calculated as follows.

$$\begin{aligned}&\int_V [\frac{\partial}{\partial\boldsymbol{B}}F(T,\boldsymbol{B})\cdot\delta\boldsymbol{B} - \boldsymbol{H}\cdot\delta\boldsymbol{B}]dV \\ &= \int_V dV\delta\boldsymbol{B}\cdot[\frac{\partial}{\partial\boldsymbol{B}}F(T,\boldsymbol{B}) - \boldsymbol{H}]dV\end{aligned} \tag{10}$$

When the variations are represented by vector potentials,

$$\delta\boldsymbol{B} = \delta\mathrm{rot}\boldsymbol{A} = \mathrm{rot}\delta\boldsymbol{A} \tag{11}$$

The above left-hand side can be further transformed as follows through partial integration using Gauss' integral theorem.

$$\begin{aligned}&\int_V dV\mathrm{rot}\delta\boldsymbol{A}\cdot[\frac{\partial}{\partial\boldsymbol{B}}F(T,\boldsymbol{B}) - \boldsymbol{H}]dV \\ &= \int_V dV\delta\boldsymbol{A}\cdot\mathrm{rot}[\frac{\partial}{\partial\boldsymbol{B}}F(T,\boldsymbol{B}) - \boldsymbol{H}]dV + \text{(Surface integral)}\end{aligned} \tag{12}$$

Since the terms of the surface integral can be eliminated by appropriate boundary conditions, the above equation becomes as follows.

$$\int_V dV\delta\boldsymbol{A}\cdot[\mathrm{rot}\frac{\partial}{\partial\boldsymbol{B}}F(T,\boldsymbol{B}) - \mathrm{rot}\boldsymbol{H}]dV \tag{13}$$

Although the distribution of the magnetic field cannot be determined, the magnetic field within this integral is subject to a rotation operator, and can be converted into current density as follows.

$$\mathrm{rot}\boldsymbol{H} = \boldsymbol{J} \tag{14}$$

Therefore, the equation obtained from the variational calculus for the thermodynamic potential, which is required from thermodynamics, is as follows.

$$\int_V dV\delta\boldsymbol{A}\cdot[\mathrm{rot}\frac{\partial}{\partial\boldsymbol{B}}F(T,\boldsymbol{B}) - \boldsymbol{J}]dV = 0 \tag{15}$$

From equation (6), we can see that this formula is equivalent to the electromagnetic field analysis equation.

Here, by examining the magnetic material thermodynamically, we have shown that magnetization characteristics can be expressed by thermodynamic potentials such as free energy, and that the variational calculus can be used in the finite element method for the electromagnetic field.